

A DEFINITE RECURSIVE RELATION AND SOME STATISTICAL PROPERTIES FOR MÖBIUS FUNCTION

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ABSTRACT. An elementary definite recursive relation for Möbius function $\mu(n)$ is introduced by two simple ways. With this recursive relation, $\mu(n)$ can be calculated without directly knowing the factorization of the n . $\mu(1) \sim \mu(2 \times 10^7)$ are calculated recursively one by one. Based on these 2×10^7 samples, the empirical probabilities of $\mu(n)$ of taking $-1, 0$, and 1 in classic statistics are calculated and compared with the theoretical probabilities in number theory. The numerical consistency between these two kinds of probability show that $\mu(n)$ could be seen as an independent random sequence when n is large. The expectation and variance of the $\mu(n)$ are 0 and $6n/\pi^2$, respectively. Furthermore, we show that any conjecture of the Mertens type is false in probability sense, and present an upper bound for cumulative sums of $\mu(n)$ with a certain probability.

Keywords:

Möbius function Recursive relation Probability sense Mertens conjecture

1. INTRODUCTION

The Möbius function is defined for a positive integer n by

$$(1) \quad \mu(n) = \begin{cases} 1 & n = 1 \\ 0 & \text{if } n \text{ is divisible by a prime square} \\ (-1)^k & \text{if } n \text{ is the product of } k \text{ distinct primes} \end{cases}$$

It is shown that Möbius function and its associated Möbius transform are important for solving different mathematical and/or scientific problems (eg., Schroeder, 2008). In physics, the Möbius function and its associated Möbius transform are used in inverse black body radiation problem (eg., Chen, 1987; 1990), inversion of specific heat data for phonon densities of states (eg., Chen et al., 1990), solution of integral equations regarding Fermi and Bose systems (eg., Chen, 2010), inverse transmissivity problem (Ji et al., 2006), and so on. All of these studies are related to how to calculate $\mu(n)$ if special methods are not used.

To calculate the Möbius function, many algorithms are presented and most of them base on the factorization of its argument. A famous one is vectorized sieving (eg., Lioen and Lune, 1994; Kuznetsov, 2011). On the other hand, in their book, Hardy and Wright (2008) showed that the Möbius function is the sum of the primitive n -th roots of unity, ie.,

$$(2) \quad \mu(n) = \sum_{\substack{1 \leq k \leq n \\ \gcd(k, n) = 1}} \exp\left(\frac{2\pi i k}{n}\right)$$

Formula (2) can be used to calculate the Möbius function without knowing the factorization of n . However, the computational complexity is not low.

Here we introduce a definite recursive relation to calculate the Möbius function without directly knowing the factorization of n as Formula (2) does, but the computational complexity is less. We calculate Möbius function from $\mu(1)$ to $\mu(2 \times 10^7)$ with this recursive relation and discuss some statistical properties of the Möbius function.

2. A DEFINITE RECURSIVE RELATION FOR MÖBIUS FUNCTION

A definite recursive relation for Möbius function can be introduced by two simple ways. One is from Möbius transform, and the other is from the Redheffer Matrix related to Mertens function which is the cumulative sum of the Möbius function. The more general relation for (poset) Möbius function can be found in the Incidence Algebra (eg., https://en.wikipedia.org/wiki/Incidence_algebra).

2.1. The recursive relation from Möbius transform. According to pair potential model for cohesive energy (Chen, 1994), the cohesive energy E for each atom in an infinite linear chain can be expressed as a sum of pairwise potentials,

$$(3) \quad E(x) = \sum_{n=1}^{\infty} \Phi(nx)$$

Using Chen-Möbius formula (e.g., Chen, 2010; Wang, 2013),

$$(4) \quad \Phi(x) = \sum_{n=1}^{\infty} \mu(n)E(nx)$$

We can write the following Matrix equality according to expression (3) and (4),

$$(5) \quad \begin{pmatrix} E(x) \\ E(2x) \\ E(3x) \\ E(4x) \\ E(5x) \\ E(6x) \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 0 & 1 & 0 & 1 & 0 & 1 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \Phi(x) \\ \Phi(2x) \\ \Phi(3x) \\ \Phi(4x) \\ \Phi(5x) \\ \Phi(6x) \\ \vdots \end{pmatrix}$$

$$(6) \quad \begin{pmatrix} \Phi(x) \\ \Phi(2x) \\ \Phi(3x) \\ \Phi(4x) \\ \Phi(5x) \\ \Phi(6x) \\ \vdots \end{pmatrix} = \begin{pmatrix} \mu(1) & \mu(2) & \mu(3) & \mu(4) & \mu(5) & \mu(6) & \cdots \\ 0 & \mu(1) & 0 & \mu(2) & 0 & \mu(3) & \cdots \\ 0 & 0 & \mu(1) & 0 & 0 & \mu(2) & \cdots \\ 0 & 0 & 0 & \mu(1) & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \mu(1) & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \mu(1) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} E(x) \\ E(2x) \\ E(3x) \\ E(4x) \\ E(5x) \\ E(6x) \\ \vdots \end{pmatrix}$$

Let

$$\Phi = [\Phi(x) \quad \Phi(2x) \quad \Phi(3x) \quad \Phi(4x) \quad \Phi(5x) \quad \Phi(6x) \quad \dots]^T$$

and

$$E = [E(x) \quad E(2x) \quad E(3x) \quad E(4x) \quad E(5x) \quad E(6x) \quad \dots]^T$$

Matrix equality (5) and (6) can be rewritten as the following,

$$(7) \quad E = U\Phi$$

$$(8) \quad \Phi = VE$$

Obviously,

$$(9) \quad UV = VU = V^T U^T = I$$

Hence the values of the Möbius function, which are the elements of the first row of the matrix V , can be obtained from the inverse matrix of U . Because the matrix U is a triangular one, in which $U = \{u_{ij}\}$ with $u_{ij} = 1$ if and only if $i=j$, one can get,

$$(10) \quad v_{1i} = - \sum_{k=1}^{i-1} v_{1k} u_{ki}, i = 2, 3, \dots$$

Based on recursive relation (10), we can obtain the recursive relation for Möbius function as the following,

$$(11) \quad \mu(n) = - \sum_{k=1}^{n-1} l_{nk} \mu(k), n = 2, 3, \dots; l_{nk} = \begin{cases} 1 & k|n \\ 0 & \text{else} \end{cases}$$

2.2. The recursive relation from Redheffer Matrix. It is well known that the Mertens function, which is the cumulative sum of the Möbius function, is the determinant of the Redheffer matrix. The Redheffer matrix $R = \{r_{ij}\}$ is defined by $r_{ij} = 1$ if $j = 1$ or $i|j$, and $r_{ij} = 0$ otherwise, ie.,

$$(12) \quad R = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 1 & 0 & 1 & 0 & 1 & \cdots \\ 1 & 0 & 1 & 0 & 0 & 1 & \cdots \\ 1 & 0 & 0 & 1 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 1 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

R can be decomposed as follows:

$$(13) \quad R = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} + \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 0 & 1 & 0 & 1 & 0 & 1 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = S + U$$

where $S = \{s_{ij}\} = 1$ if and only if $j = 1$ and $i \neq 1$; U is the same to matrix equality (7) with $u_{ij} = 1$ if and only if $i|j$.

It can be shown that the inverse of U is V which is in matrix equality (8), that is,

$$(14) \quad V = \{v_{ij}\} = \begin{cases} \mu(\frac{j}{i}) & i|j \\ 0 & \text{else} \end{cases}$$

In fact, the ij -th entry of the product of $U \times V$, p_{ij} , is,

$$(15) \quad p_{ij} = \sum_{k=1}^n u_{ik} v_{kj}$$

According to the definition of U and V in (14), $u_{ik} v_{kj}$ is 0 unless $i|k$ and $k|j$, which means that $p_{ij} = 0$ if $i \nmid j$. If $i|j$, by using the well known

$$\sum_{i|n} \mu(i) = \begin{cases} 1 & n = 1 \\ 0 & \text{else} \end{cases}$$

one can get,

$$p_{ij} = \sum_{k(i|k \text{ and } k|j)} \mu\left(\frac{j}{k}\right) = \sum_{k'(j|i)} \mu\left(\frac{j/i}{k'}\right) = \begin{cases} 1 & j = i \\ 0 & \text{else} \end{cases}$$

Therefore, $U \times V = \{p_{ij}\} = I$ and equality (14) holds. With the same procedures in the subsection above, we can obtain the recursive relation (11) for the Möbius function.

3. SOME STATISTICAL PROPERTIES FOR MÖBIUS FUNCTION

With the recursive relation (11), we calculated the Möbius function from $\mu(1)$ to $\mu(2 \times 10^7)$. These values are used for the numerical test on some statistical properties of Möbius sequence $\mu(n)$, if $\mu(n)$ is seen as an independent random sequence although it has a deterministic recursive rule. In fact, as n is large enough, the random assumption above is reasonable.

3.1. The expectation and variance of the $\mu(n)$. Firstly we calculated the probabilities of $\mu(n)$ of taking the values -1 , 0 and 1 . In this respect, there are two useful results from Hardy and Wright (2008) as follows,

1. $\mu(n) = \pm 1$ or $|\mu(n)| = 1$ if a number n is squarefree, and the probability (p_t) that a number should be squarefree is $\frac{6}{\pi^2}$, more precisely,

$$(16) \quad \sum_{n=1}^x |\mu(n)| = \frac{6}{\pi^2}x + O(\sqrt{x})$$

2. Among the squarefree numbers, those for $\mu(n) = 1$ and those for $\mu(n) = -1$ occur with about the same frequency.

Therefore, if $\mu(k)$ ($k = 1, 2, \dots, n$) denotes the k -th value of $\mu(n)$ and p_{t_k} the probability, the corresponding distribution rule is shown in Table 1.

Table 1: The distribution rule for $\mu(k)$

$\mu(k)$	-1	0	1
p_{t_k}	$3/\pi^2$	$1 - 6/\pi^2$	$3/\pi^2$

The p_t is different from that in the classic statistics (See details in Hardy and Wright (2008), P. 354). To use the methods in classic statistic, it is necessary firstly to test the consistency between them. The classic probability here is,

$$(17) \quad p_e = \frac{N_{\mu(n)=m}}{n}, (m = -1, 0, 1)$$

where p_e is the classic probability that $\mu(n) = m$ ($m = -1, 0, 1$), $N_{\mu(n)=m}$ is the frequency of $\mu(n) = m$.

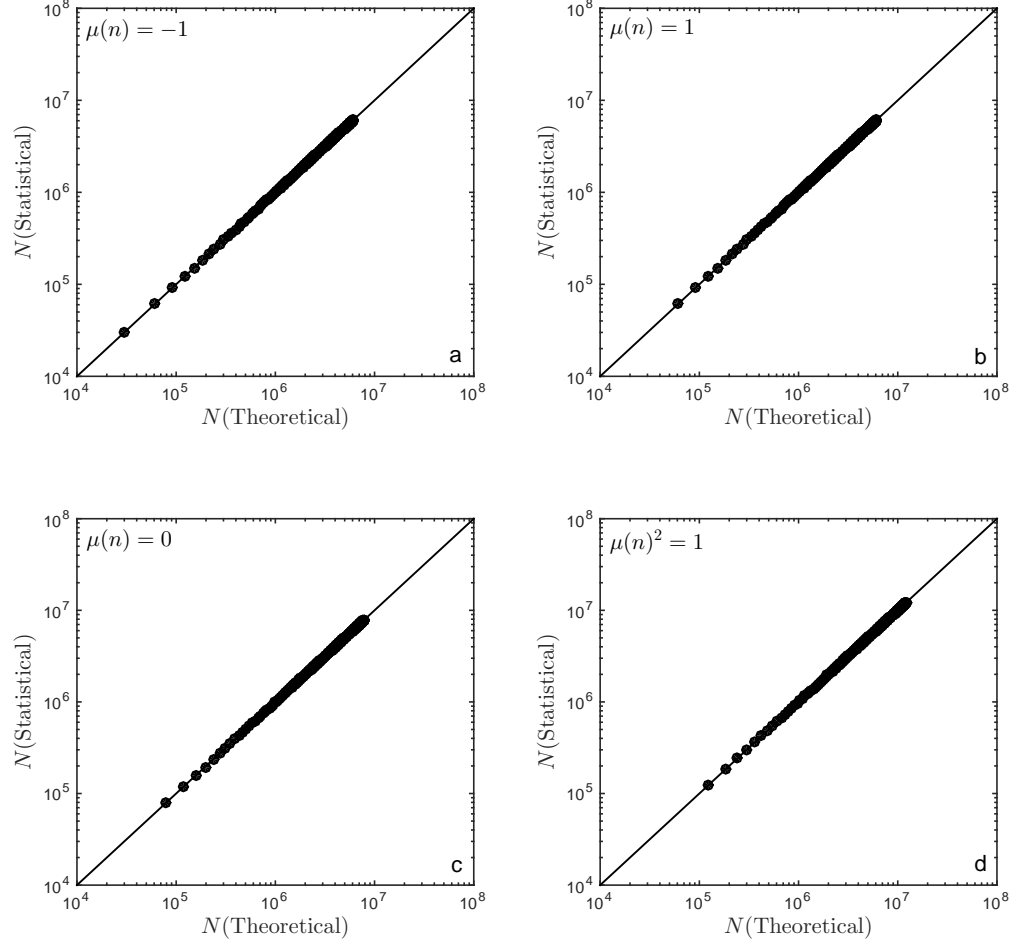


Fig. 1: The comparison of frequencies observed ($N(\text{Statistical})$) with those calculated by number theory ($N(\text{Theoretical})$) in blocks with different length of N . The solid line in each subfigure is the reference line. The related data are shown in Table 3 and 4 in the appendix.

We calculate the frequencies and p_e s of $\mu(n)$ of taking $-1, 0, 1$ and that of $|\mu(n)| = \mu^2(n) = 1$ in 200 blocks with different length by using 2×10^7 $\mu(n)$ s above, respectively. Figure 1 shows the comparison of these frequencies observed with those calculated by $N \times p_t$ in different blocks of length N . It can be seen that the frequencies observed are consistent with those calculated. Figure 2 shows the numerical comparison of classic probability p_e with the p_t . It also can be observed that these two kinds of probabilities are numerically

consistent. Detail numerical results are shown in Table 3 and 4 in the appendix. These consistencies above show that the p_t is equivalent numerically to the classic probability p_e as defined in (17). Similar numerical support can be found in Good and Churchhouse (1968). Based on these numerical results, we can take $\mu(n)$ as an independent random sequence although it has a deterministic recursive rule and use classic statistical method to study it.

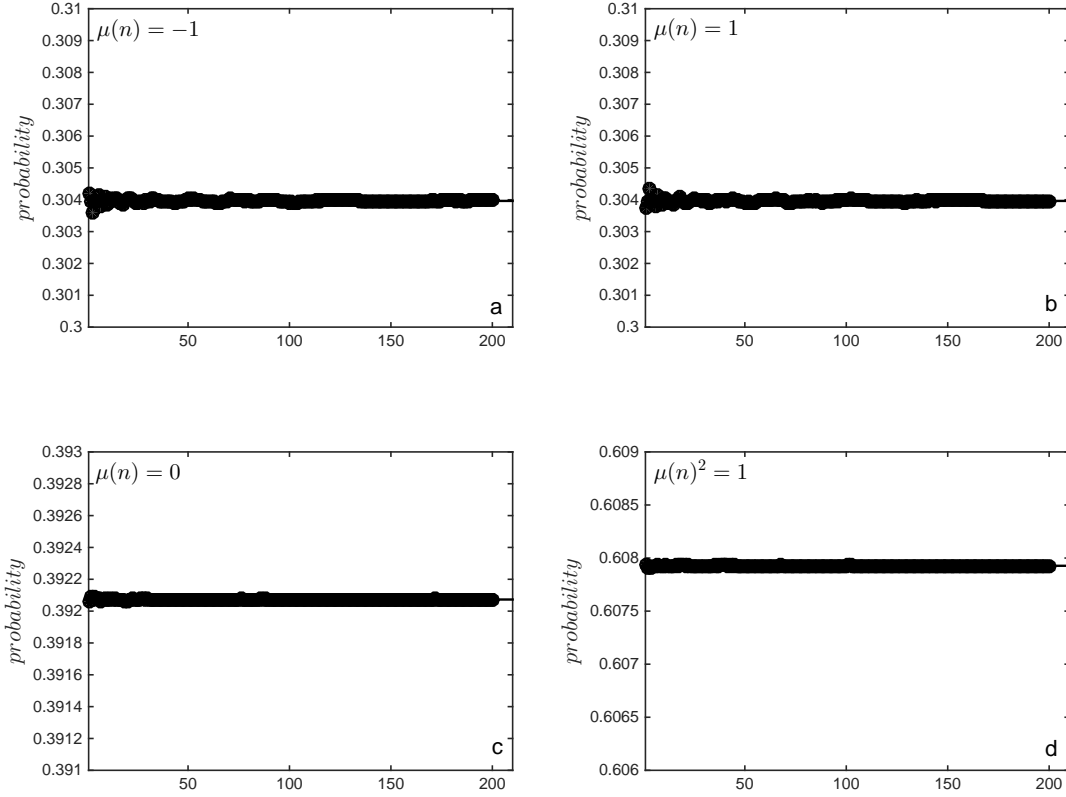


Fig. 2: The comparison of classic probability p_e (solid dots) with the p_t (solid lines) in blocks with different length of N . In each subfigure, the x-axis is ordinal number of each block. The related data are shown in Table 3 and 4 in the appendix.

Accordingly, the expectation and variance of the $\mu(k)$ are $E(\mu(k)) = 0$ and $D(\mu(k)) = 6/\pi^2$ from Table 1, respectively. These results are consistent with the conjecture of Good and Churchhouse (1968). The expectation and variance of the $\mu(k)$ will be used in the following section.

3.2. Mertens conjecture in a statistical point of view. Mertens function of a positive integer n is defined as the cumulative sums of $\mu(n)$,

$$(18) \quad M(n) = \sum_{k=1}^n \mu(k)$$

An old conjecture, "Mertens conjecture", proposed that $|M(n)| < n^{1/2}$ for all n . This was disproved by Odlyzko and te Riele (1985). In this subsection, we recheck Mertens conjecture in a statistical point of view, for $\mu(n)$ is seen as an independent random sequence although it has a deterministic recursive rule.

According to central limit theorem, if n is large enough, for any x , we have,

$$(19) \quad \lim_{n \rightarrow \infty} P \left\{ \frac{\sum_{k=1}^n \mu(k) - E(\sum_{k=1}^n \mu(k))}{\sqrt{D(\sum_{k=1}^n \mu(k))}} \leq x \right\} = \lim_{n \rightarrow \infty} P \left\{ \frac{M(n)}{\sqrt{6n/\pi^2}} \leq x \right\} \\ = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} \exp(-\frac{t^2}{2}) dt \\ = \Phi(x)$$

where, $E(\sum_{k=1}^n \mu(k)) = 0$, $D(\sum_{k=1}^n \mu(k)) = 6n/\pi^2$ according to subsection 3.1. And (19) means,

$$(20) \quad \frac{M(n)}{\sqrt{6n/\pi^2}} \sim N(0, 1)$$

as n is large.

Figure 3 shows the probability density function for $\frac{M(n)}{\sqrt{6n/\pi^2}}$ when $n = 500000$. It can be observed that the distribution of (20) is reasonable. Another similar numerical support for this can be found in Good and Churchhouse (1968).

With equality (19), the probability of $M(n) > \sqrt{n}$ can be obtained. Clearly,

$$(21) \quad P \{M(n) > \sqrt{n}\} = 1 - \Phi\left(\frac{1}{\sqrt{6/\pi^2}}\right) \approx 0.0998$$

That is, the probability of $M(n) > \sqrt{n}$ is about 0.0998 but not 0, which means that Mertens conjecture is not true. Furthermore, any conjecture of the Mertens type, viz.

$$(22) \quad |M(n)| < C\sqrt{n}$$

where C is any positive constant, is false, unless C is large enough.

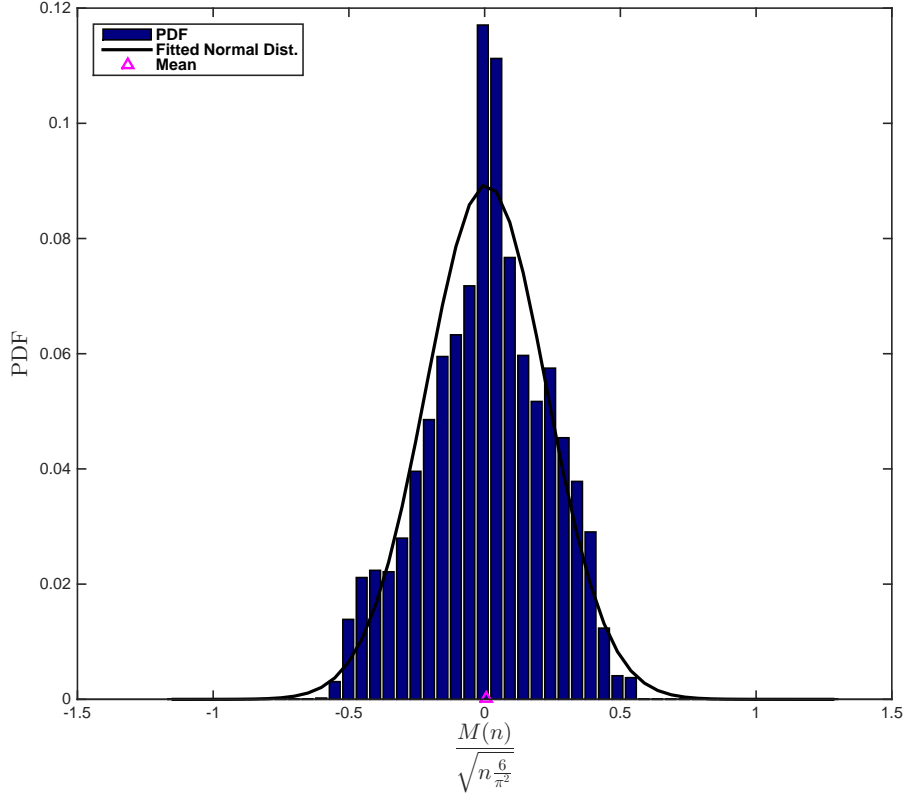


Fig. 3: The probability density function for $\frac{M(n)}{\sqrt{6n/\pi^2}}$ when $n = 500000$.

3.3. Upper bound of cumulative sums of $\mu(n)$ sequence. From (19), one can get when n is large,

$$(23) \quad \frac{\sum_{k=1}^n \mu(k) - E\left(\sum_{k=1}^n \mu(k)\right)}{\sqrt{D\left(\sum_{k=1}^n \mu(k)\right)}} = \frac{\sum_{k=1}^n \mu(k) - nu}{\sqrt{n}\sigma} \sim N(0, 1)$$

Then a confidence interval for expectation u with a known standard variance σ and a probability of $1 - \alpha$ is,

$$(24) \quad \left[-\frac{\sigma}{\sqrt{n}} K_{\alpha/2}, \frac{\sigma}{\sqrt{n}} K_{\alpha/2} \right]$$

where,

$$(25) \quad \int_{-K_{\alpha/2}}^{K_{\alpha/2}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt = 1 - \alpha$$

From (24) and (25), one can infer further that the upper bound of $\sum_{k=1}^n \mu(k)$, with $u = 0$ and $\sigma = \sqrt{6/\pi^2}$ for $\mu(k)$, is,

$$(26) \quad \sum_{k=1}^n \mu(k) = M(n) \leq \sigma K_{\alpha/2} \sqrt{n} = \sqrt{6/\pi^2} K_{\alpha/2} \sqrt{n}$$

The inequality (26) holds with a probability of $1 - \alpha$.

In fact, the inequality of Mertens type is only a special case of (26) with a fixed probability of $1 - \alpha$.

4. DISCUSSIONS

4.1. The calculation of $\mu(n)$ with the recursive relation. In theory, we can calculate $\mu(n)$ for any large n recursively with the recursive relation obtained in section 2. However, in order to calculate $\mu(n)$, we need to know $\mu(1), \mu(2), \mu(3), \dots, \mu(n-1)$. Usually $\mu(1), \mu(2), \mu(3), \dots, \mu(n-1)$ are stored in an array which demands much larger amount of computer memory if n is large. In this paper, we only calculate the values of $\mu(n)$ from $\mu(1)$ to $\mu(2 \times 10^7)$ because of the memory limitation of our desktop computer and computing time. To obtain more numerical results of $\mu(n)$ with large n , both the faster and/or optimization algorithm for the recursive relation here and better hardware platform are required. It is a probable way by which the calculations are divided into blocks and are computed with GPU, or quantum computer will be used in the future.

4.2. The independent randomness of $\mu(n)$. Based on the numerical consistency between empirical statistical quantities for only 2×10^7 $\mu(n)$ and those from number theory (eg., $N(\text{Statistical})$) and $N(\text{Theoretical})$, p_e and p_t), we use classic statistical method to study $\mu(n)$ regardless of the strict validity of the independent randomness of $\mu(n)$. In the respect of the independent randomness of $\mu(n)$, there are some discussions (eg., Sarnak, 2012). Although $\mu(n)$ is deterministic from the recursive relation in section 2, it is visually random and independent. Figure 4 shows that the variation of $\sum_{k=1}^n \mu(k)/n$ with n when $n = 500000$. It can be observed that $\mu(n)$ has some properties of the independent random variable.

The above can be viewed restrictedly from another definition of $\mu(n)$,

$$(27) \quad \mu(n) = \begin{cases} 0 & \text{if } n \text{ is non-squarefree} \\ (-1)^{\omega(n)} & \text{if } n \text{ is squarefree} \end{cases}$$

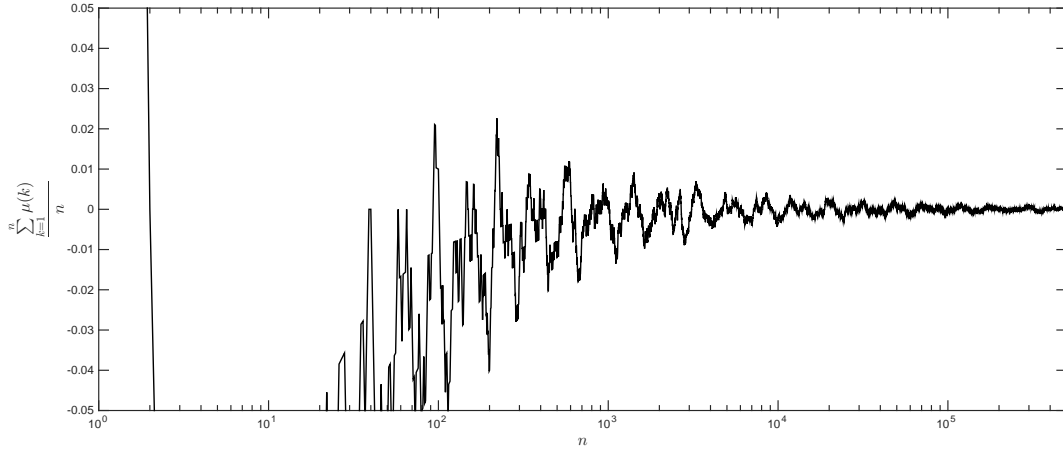


Fig. 4: Variation of $\frac{\sum_{k=1}^n \mu(k)}{n}$ with n when $n = 500000$.

where $\omega(n)$ is the number of distinct prime factors.

According to Erdős-Kac Theorem, $\omega(n)$ is independent and random when n is large, so may be $\mu(n)$ for $\mu(n) = (-1)^{\omega(n)} = \exp[iz\omega(n)]|_{z=\pi}$ when n is squarefree and if we take n in random order.

Furthermore, $\mu(n)$ is either $(-1)^{\omega(n)}$ or 0 because of its definition of (27), so we have $M(n) = M(n') = \sum_{k=1}^{n'} \mu(k) = \sum_{k=1}^{n'} (-1)^{\omega(k)}$ (n' is squarefree). The randomness of $M(n')$ should be stronger than $\mu(n)$ if we take n' in random order. Numerical results of Good and Churchhouse (1968) show $M(n)$ in blocks of length N has asymptotically a normal distribution with mean zero and variance of $6N/\pi^2$ (where N is large). These numerical results can be rechecked as follows. From the rules in mathematical statistics, we know that the observed values of a discrete random variable X ($X = x_1, x_2, \dots, x_n$) lie in the following interval with probability $p > 1 - \alpha$, for real number α with $0 < \alpha < 1$,

$$(28) \quad \left[\bar{X} - \frac{\Delta X}{\sqrt{\alpha}}, \bar{X} + \frac{\Delta X}{\sqrt{\alpha}} \right]$$

where $\bar{X} = \sum_{k=1}^n x_k p_k$, $(\Delta X)^2 = \sum_{k=1}^n (x_k - \bar{X})^2 p_k$, $p_k := P(x = x_k)$.

Obviously, for $M(n) = M(n') = \sum_{k=1}^{n'} (-1)^{\omega(k)}$, $p_k = 6/\pi^2$ according to subsection (3.1), and further one have,

$$\begin{aligned}\bar{X} &= \frac{6}{\pi^2} \sum_{k=1}^{n'_{\text{even}}} 1 + \frac{6}{\pi^2} \sum_{l=1}^{n'_{\text{odd}}} -1 = 0 \\ (\Delta X)^2 &= \frac{6}{\pi^2} \sum_{k=1}^{n'_{\text{even}}} 1 + \frac{6}{\pi^2} \sum_{l=1}^{n'_{\text{odd}}} 1 = \frac{6n'}{\pi^2}\end{aligned}$$

where n'_{even} is squarefree with even number of distinct prime factors, n'_{odd} with odd number of distinct prime factors.

Therefore, from (28), we can obtain an upper bound for $M(n)$ similar to (26) as follows, with probability $p > 1 - \alpha$,

$$(29) \quad \sum_{k=1}^n \mu(k) = M(n) = \sum_{k=1}^{n'} (-1)^{\omega(k)} \leq \frac{\sqrt{6/\pi^2}}{\sqrt{\alpha}} \sqrt{n'} \leq \frac{\sqrt{6/\pi^2}}{\sqrt{\alpha}} \sqrt{n}$$

If α takes $\frac{6}{\pi^2}$, then $M(n) \leq \sqrt{n}$ with a probability $p > 1 - 6/\pi^2 \approx 0.3920$, which means that Mertens conjecture is not true.

On the other hand, we can check whether $\mu(n)$ is periodic or not by estimating its power spectral density (PSD). We calculate the PSD for $\mu(n)$ series from $\mu(1)$ to $\mu(2 \times 10^7)$ by taking n as time. The results are shown in Figure 5. It can be found that $\mu(n)$ has no apparent periodicity because the PSD of $\mu(n)$ have no distinguished peak(s).

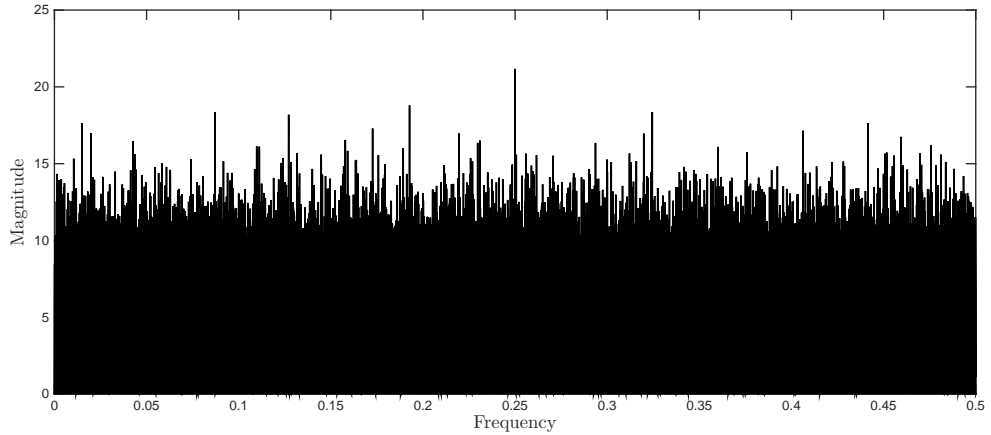


Fig. 5: The power spectral density (PSD) for $\mu(n)$ series from $\mu(1)$ to $\mu(2 \times 10^7)$.

Although the independent randomness of $\mu(n)$ is a problem unsolved so far, we can analyse $\mu(n)$ by way of statistics because $\mu(n)$ has a complicate and non-periodic distribution, as those statistical approaches applied to chaos.

4.3. **The probability of $\sum_{k=1}^n |\mu(k)| > Cn$.** Similarly, we can calculate the probability of

$\sum_{k=1}^n |\mu(k)| > Cn$. The distribution law for $|\mu(n)|$ can be obtained as shown in Table 2.

Table 2: The distribution rule for $|\mu(k)|$

$ \mu(k) $	0	1
p_{t_k}	$1 - 6/\pi^2$	$6/\pi^2$

And $E(|\mu(k)|) = 6/\pi^2$ and $D(|\mu(k)|) = 6/\pi^2(1 - 6/\pi^2)$.
According to central limit theorem, for any x , we have,

$$(30) \quad \lim_{n \rightarrow \infty} P \left\{ \frac{\sum_{k=1}^n |\mu(k)| - E(\sum_{k=1}^n |\mu(k)|)}{\sqrt{D(\sum_{k=1}^n |\mu(k)|)}} \leq x \right\} = \lim_{n \rightarrow \infty} P \left\{ \frac{\sum_{k=1}^n |\mu(k)| - 6n/\pi^2}{\sqrt{n6/\pi^2(1-6/\pi^2)}} \leq x \right\}$$

$$= \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} \exp(-\frac{t^2}{2}) dt$$

$$= \Phi(x)$$

And (30) means,

$$(31) \quad \frac{\sum_{k=1}^n |\mu(k)| - 6n/\pi^2}{\sqrt{n6/\pi^2(1-6/\pi^2)}} \sim N(0, 1)$$

Figure 6 shows the probability density function for $\frac{\sum_{k=1}^n |\mu(k)| - 6n/\pi^2}{\sqrt{n6/\pi^2(1-6/\pi^2)}}$ when $n = 500000$. It can be seen that the distribution of (31) is reasonable.

With equality (30), the probability of $\sum_{k=1}^n |\mu(k)| > Cn$ (where C is a constant) can be obtained. Clearly,

$$(32) \quad P \left\{ \sum_{k=1}^n |\mu(k)| > Cn \right\} = P \left\{ \frac{\sum_{k=1}^n |\mu(k)| - 6n/\pi^2}{\sqrt{n6/\pi^2(1-6/\pi^2)}} > \frac{Cn - 6n/\pi^2}{\sqrt{n6/\pi^2(1-6/\pi^2)}} \right\}$$

$$= 1 - P \left\{ \frac{\sum_{k=1}^n |\mu(k)| - 6n/\pi^2}{\sqrt{n6/\pi^2(1-6/\pi^2)}} \leq \frac{Cn - 6n/\pi^2}{\sqrt{n6/\pi^2(1-6/\pi^2)}} \right\}$$

$$= 1 - \Phi \left\{ \frac{(C - 6/\pi^2)n}{\sqrt{n6/\pi^2(1-6/\pi^2)}} \right\}$$

With (32), when n is large

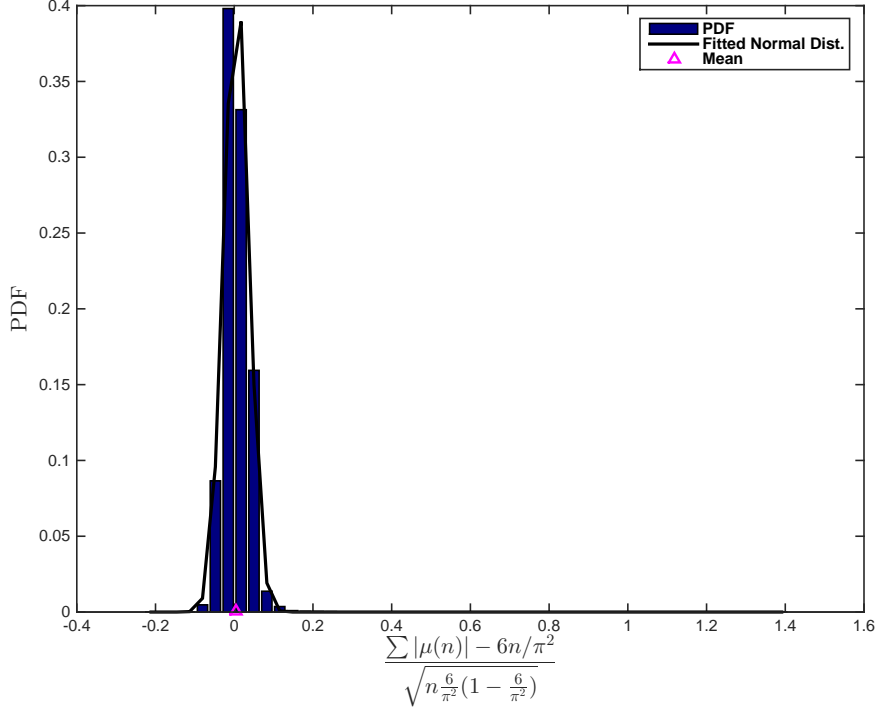


Fig. 6: The probability density function for $\frac{\sum_{k=1}^n |\mu(k)| - 6n/\pi^2}{\sqrt{n \frac{6}{\pi^2} (1 - \frac{6}{\pi^2})}}$ when $n = 500000$.

$$(33) \quad P \left\{ \sum_{k=1}^n |\mu(k)| > Cn \right\} = \begin{cases} 0 & C > 6/\pi^2 \\ 1/2 & C = 6/\pi^2 \\ 1 & C < 6/\pi^2 \end{cases}$$

5. CONCLUSIONS

Based on the results and discussion above, some conclusions can be drawn as follows,

(1) An elementary definite recursive relation for Möbius function is introduced by two simple ways. One is from Möbius transform, and the other is from Redheffer Matrix. With this recursive relation, $\mu(n)$ can be calculated without directly knowing the factorization of n , in which the most complex operation is only the Mod.

(2) With this relation, $\mu(1) \sim \mu(2 \times 10^7)$ are calculated recursively. Based on these 2×10^7 samples, we calculate the frequencies and empirical probabilities for $\mu(n)$ of taking $-1, 0, 1$, so does for $|\mu| = 1$. And then compare them with those in number theory. It can be found these two kinds of frequencies and probabilities are numerically consistent.

(3) Based on these numerical results, we take $\mu(n)$ as an independent random sequence although it has a deterministic recursive rule. The expectation and variance of the $\mu(k)$ are $E(\mu(k)) = 0$ and $D(\mu(k)) = 6/\pi^2$, respectively.

(4) We show that the Mertens conjecture, even any conjecture of the Mertens type, is false in a probability sense, and present an upper bound for cumulative sums of $\mu(n)$ as $\sum_{k=1}^n \mu(k) \leq \sqrt{6/\pi^2} K_{\alpha/2} \sqrt{n}$ with a probability of $1 - \alpha$.

Acknowledges

We thank Russ Woodroffe very much for pointing out that the recursive relation (11) is a special case for (poset) Möbius function in Incidence Algebra.

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Appendix

Table 3: The comparison of empirical probability and frequency with those from number theory when $\mu(n) = -1/1$

N	N_{-1}	N_{-1}^T	$p_e(0.3039635509)$	N_1	N_1^T	$p_e(0.3039635509)$
100000	30421	30396.4	0.3042100000	30373	30396.4 0	0.3037300000
200000	60791	60792.7	0.3039550000	60790	60792.7 0	0.3039500000
300000	91079	91189.1	0.3035966667	91299	91189.1 0	0.3043300000
400000	121577	121585.4	0.3039425000	121588	121585.4	0.3039700000
500000	151982	151981.8	0.3039640000	151976	151981.8	0.3039520000
600000	182492	182378.1	0.3041533333	182262	182378.1	0.3037700000
700000	212666	212774.5	0.3038085714	212892	212774.5	0.3041314286
800000	243181	243170.8	0.3039762500	243161	243170.8	0.3039512500
900000	273678	273567.2	0.3040866667	273453	273567.2	0.3038366667
1000000	303857	303963.6	0.3038570000	304069	303963.6	0.3040690000
1100000	334263	334359.9	0.3038754545	334464	334359.9	0.3040581818
1200000	364832	364756.3	0.3040266667	364677	364756.3	0.3038975000
1300000	395192	395152.6	0.3039938462	395111	395152.6	0.3039315385
1400000	425669	425549.0	0.3040492857	425422	425549.0	0.3038728571
1500000	456092	455945.3	0.3040613333	455799	455945.3	0.3038660000
1600000	486262	486341.7	0.3039137500	486430	486341.7	0.3040187500
1700000	516688	516738.0	0.3039341176	516792	516738.0	0.3039952941
1800000	546936	547134.4	0.3038533333	547340	547134.4	0.3040777778
1900000	577533	577530.7	0.3039647368	577544	577530.7	0.3039705263
2000000	608062	607927.1	0.3040310000	607815	607927.1	0.3039075000
2100000	638508	638323.5	0.3040514286	638142	638323.5	0.3038771429
2200000	668846	668719.8	0.3040209091	668600	668719.8	0.3039090909
2300000	699203	699116.2	0.3040013043	699016	699116.2	0.3039200000
2400000	729384	729512.5	0.3039100000	729638	729512.5	0.3040158333
2500000	759726	759908.9	0.3038904000	760088	759908.9	0.3040352000
2600000	790230	790305.2	0.3039346154	790377	790305.2	0.3039911538
2700000	820674	820701.6	0.3039533333	820722	820701.6	0.3039711111
2800000	850937	851097.9	0.3039060714	851251	851097.9	0.3040182143
2900000	881525	881494.3	0.3039741379	881457	881494.3	0.3039506897
3000000	911833	911890.7	0.3039443333	911940	911890.7	0.3039800000
3100000	942195	942287.0	0.3039338710	942372	942287.0	0.3039909677
3200000	972925	972683.4	0.3040390625	972438	972683.4	0.3038868750
3300000	1003355	1003079.7	0.3040469697	1002803	1003079.7	0.3038796970
3400000	1033623	1033476.1	0.3040067647	1033331	1033476.1	0.3039208824
3500000	1063947	1063872.4	0.3039848571	1063809	1063872.4	0.3039454286
3600000	1094166	1094268.8	0.3039350000	1094378	1094268.8	0.3039938889
3700000	1124662	1124665.1	0.3039627027	1124661	1124665.1	0.3039624324

Table 3: The comparison of empirical probability and frequency with those from number theory when $\mu(n) = -1/1$ (continued)

N	N_{-1}	N_{-1}^T	$p_e(0.3039635509)$	N_1	N_1^T	$p_e(0.3039635509)$
3800000	1154989	1155061.5	0.3039444737	1155144	1155061.5	0.3039852632
3900000	1185390	1185457.8	0.3039461538	1185540	1185457.8	0.3039846154
4000000	1215772	1215854.2	0.3039430000	1215964	1215854.2	0.3039910000
4100000	1246259	1246250.6	0.3039656098	1246258	1246250.6	0.3039653659
4200000	1276499	1276646.9	0.3039283333	1276799	1276646.9	0.3039997619
4300000	1306851	1307043.3	0.3039188372	1307252	1307043.3	0.3040120930
4400000	1337169	1337439.6	0.3039020455	1337720	1337439.6	0.3040272727
4500000	1367757	1367836.0	0.3039460000	1367930	1367836.0	0.3039844444
4600000	1398114	1398232.3	0.3039378261	1398354	1398232.3	0.3039900000
4700000	1428655	1428628.7	0.3039691489	1428604	1428628.7	0.3039582979
4800000	1459035	1459025.0	0.3039656250	1459025	1459025.0	0.3039635417
4900000	1489629	1489421.4	0.3040059184	1489223	1489421.4	0.3039230612
5000000	1520171	1519817.8	0.3040342000	1519462	1519817.8	0.3038924000
5100000	1550492	1550214.1	0.3040180392	1549940	1550214.1	0.3039098039
5200000	1580813	1580610.5	0.3040025000	1580407	1580610.5	0.3039244231
5300000	1611343	1611006.8	0.3040269811	1610658	1611006.8	0.3038977358
5400000	1641700	1641403.2	0.3040185185	1641101	1641403.2	0.3039075926
5500000	1672051	1671799.5	0.3040092727	1671538	1671799.5	0.3039160000
5600000	1702293	1702195.9	0.3039808929	1702098	1702195.9	0.3039460714
5700000	1732416	1732592.2	0.3039326316	1732764	1732592.2	0.3039936842
5800000	1762756	1762988.6	0.3039234483	1763212	1762988.6	0.3040020690
5900000	1793193	1793385.0	0.3039310169	1793570	1793385.0	0.3039949153
6000000	1823650	1823781.3	0.3039416667	1823907	1823781.3	0.3039845000
6100000	1854028	1854177.7	0.3039390164	1854325	1854177.7	0.3039877049
6200000	1884474	1884574.0	0.3039474194	1884666	1884574.0	0.3039783871
6300000	1914862	1914970.4	0.3039463492	1915064	1914970.4	0.3039784127
6400000	1945014	1945366.7	0.3039084375	1945715	1945366.7	0.3040179687
6500000	1975328	1975763.1	0.3038966154	1976195	1975763.1	0.3040300000
6600000	2005813	2006159.4	0.3039110606	2006509	2006159.4	0.3040165152
6700000	2036280	2036555.8	0.3039223881	2036837	2036555.8	0.3040055224
6800000	2066717	2066952.1	0.3039289706	2067210	2066952.1	0.3040014706
6900000	2097200	2097348.5	0.3039420290	2097505	2097348.5	0.3039862319
7000000	2127844	2127744.9	0.3039777143	2127660	2127744.9	0.3039514286
7100000	2158600	2158141.2	0.3040281690	2157693	2158141.2	0.3039004225
7200000	2188861	2188537.6	0.3040084722	2188217	2188537.6	0.3039190278
7300000	2219265	2218933.9	0.3040089041	2218604	2218933.9	0.3039183562
7400000	2249572	2249330.3	0.3039962162	2249081	2249330.3	0.3039298649
7500000	2279804	2279726.6	0.3039738667	2279632	2279726.6	0.3039509333

Table 3: The comparison of empirical probability and frequency with those from number theory when $\mu(n) = -1/1$ (continued)

N	N_{-1}	N_{-1}^T	$p_e(0.3039635509)$	N_1	N_1^T	$p_e(0.3039635509)$
7600000	2310212	2310123.0	0.3039752632	2310014	2310123.0	0.3039492105
7700000	2340655	2340519.3	0.3039811688	2340371	2340519.3	0.3039442857
7800000	2370879	2370915.7	0.3039588462	2370945	2370915.7	0.3039673077
7900000	2401416	2401312.1	0.3039767089	2401202	2401312.1	0.3039496203
8000000	2431796	2431708.4	0.3039745000	2431607	2431708.4	0.3039508750
8100000	2462041	2462104.8	0.3039556790	2462157	2462104.8	0.3039700000
8200000	2492428	2492501.1	0.3039546341	2492563	2492501.1	0.3039710976
8300000	2522880	2522897.5	0.3039614458	2522900	2522897.5	0.3039638554
8400000	2553329	2553293.8	0.3039677381	2553250	2553293.8	0.3039583333
8500000	2583583	2583690.2	0.3039509412	2583782	2583690.2	0.3039743529
8600000	2614150	2614086.5	0.3039709302	2614001	2614086.5	0.3039536047
8700000	2644785	2644482.9	0.3039982759	2644163	2644482.9	0.3039267816
8800000	2675324	2674879.2	0.3040140909	2674412	2674879.2	0.3039104545
8900000	2705512	2705275.6	0.3039901124	2705036	2705275.6	0.3039366292
9000000	2735841	2735672.0	0.3039823333	2735501	2735672.0	0.3039445556
9100000	2766275	2766068.3	0.3039862637	2765847	2766068.3	0.3039392308
9200000	2796640	2796464.7	0.3039826087	2796280	2796464.7	0.3039434783
9300000	2827009	2826861.0	0.3039794624	2826712	2826861.0	0.3039475269
9400000	2857367	2857257.4	0.3039752128	2857156	2857257.4	0.3039527660
9500000	2887605	2887653.7	0.3039584211	2887723	2887653.7	0.3039708421
9600000	2918034	2918050.1	0.3039618750	2918074	2918050.1	0.3039660417
9700000	2948319	2948446.4	0.3039504124	2948598	2948446.4	0.3039791753
9800000	2978563	2978842.8	0.3039350000	2979144	2978842.8	0.3039942857
9900000	3008889	3009239.2	0.3039281818	3009605	3009239.2	0.3040005051
10000000	3039127	3039635.5	0.3039127000	3040164	3039635.5	0.3040164000
10100000	3069687	3070031.9	0.3039294059	3070405	3070031.9	0.3040004950
10200000	3099872	3100428.2	0.3039090196	3101015	3100428.2	0.3040210784
10300000	3130336	3130824.6	0.3039161165	3131328	3130824.6	0.3040124272
10400000	3160738	3161220.9	0.3039171154	3161719	3161220.9	0.3040114423
10500000	3191206	3191617.3	0.3039243810	3192048	3191617.3	0.3040045714
10600000	3221686	3222013.6	0.3039326415	3222348	3222013.6	0.3039950943
10700000	3252202	3252410.0	0.3039441121	3252627	3252410.0	0.3039838318
10800000	3282578	3282806.4	0.3039424074	3283049	3282806.4	0.3039860185
10900000	3312999	3313202.7	0.3039448624	3313429	3313202.7	0.3039843119
11000000	3343506	3343599.1	0.3039550909	3343712	3343599.1	0.3039738182
11100000	3374011	3373995.4	0.3039649550	3373989	3373995.4	0.3039629730
11200000	3404419	3404391.8	0.3039659821	3404376	3404391.8	0.3039621429
11300000	3434879	3434788.1	0.3039715929	3434711	3434788.1	0.3039567257

Table 3: The comparison of empirical probability and frequency with those from number theory when $\mu(n) = -1/1$ (continued)

N	N_{-1}	N_{-1}^T	$p_e(0.3039635509)$	N_1	N_1^T	$p_e(0.3039635509)$
11400000	3465185	3465184.5	0.3039635965	3465208	3465184.5	0.3039656140
11500000	3495628	3495580.8	0.3039676522	3495538	3495580.8	0.3039598261
11600000	3526180	3525977.2	0.3039810345	3525787	3525977.2	0.3039471552
11700000	3556685	3556373.5	0.3039901709	3556070	3556373.5	0.3039376068
11800000	3587067	3586769.9	0.3039887288	3586490	3586769.9	0.3039398305
11900000	3617377	3617166.3	0.3039812605	3616976	3617166.3	0.3039475630
12000000	3647897	3647562.6	0.3039914167	3647243	3647562.6	0.3039369167
12100000	3678249	3677959.0	0.3039875207	3677691	3677959.0	0.3039414050
12200000	3708727	3708355.3	0.3039940164	3707994	3708355.3	0.3039339344
12300000	3738956	3738751.7	0.3039801626	3738560	3738751.7	0.3039479675
12400000	3769655	3769148.0	0.3040044355	3768649	3769148.0	0.3039233065
12500000	3800038	3799544.4	0.3040030400	3799058	3799544.4	0.3039246400
12600000	3830284	3829940.7	0.3039907937	3829590	3829940.7	0.3039357143
12700000	3860722	3860337.1	0.3039938583	3859952	3860337.1	0.3039332283
12800000	3891316	3890733.5	0.3040090625	3890157	3890733.5	0.3039185156
12900000	3921803	3921129.8	0.3040157364	3920466	3921129.8	0.3039120930
13000000	3952009	3951526.2	0.3040006923	3951058	3951526.2	0.3039275385
13100000	3982303	3981922.5	0.3039925954	3981547	3981922.5	0.3039348855
13200000	4012618	4012318.9	0.3039862121	4012019	4012318.9	0.3039408333
13300000	4042856	4042715.2	0.3039741353	4042567	4042715.2	0.3039524060
13400000	4073262	4073111.6	0.3039747761	4072950	4073111.6	0.3039514925
13500000	4103423	4103507.9	0.3039572593	4103593	4103507.9	0.3039698519
13600000	4133881	4133904.3	0.3039618382	4133938	4133904.3	0.3039660294
13700000	4164370	4164300.6	0.3039686131	4164238	4164300.6	0.3039589781
13800000	4194724	4194697.0	0.3039655072	4194671	4194697.0	0.3039616667
13900000	4225100	4225093.4	0.3039640288	4225097	4225093.4	0.3039638129
14000000	4255463	4255489.7	0.3039616429	4255509	4255489.7	0.3039649286
14100000	4285549	4285886.1	0.3039396454	4286231	4285886.1	0.3039880142
14200000	4315867	4316282.4	0.3039342958	4316712	4316282.4	0.3039938028
14300000	4346401	4346678.8	0.3039441259	4346971	4346678.8	0.3039839860
14400000	4377024	4377075.1	0.3039600000	4377125	4377075.1	0.3039670139
14500000	4407257	4407471.5	0.3039487586	4407682	4407471.5	0.3039780690
14600000	4437734	4437867.8	0.3039543836	4438008	4437867.8	0.3039731507
14700000	4467979	4468264.2	0.3039441497	4468549	4468264.2	0.3039829252
14800000	4498441	4498660.6	0.3039487162	4498892	4498660.6	0.3039791892
14900000	4528728	4529056.9	0.3039414765	4529378	4529056.9	0.3039851007
15000000	4559112	4559453.3	0.3039408000	4559777	4559453.3	0.3039851333
15100000	4589543	4589849.6	0.3039432450	4590147	4589849.6	0.3039832450

Table 3: The comparison of empirical probability and frequency with those from number theory when $\mu(n) = -1/1$ (continued)

N	N_{-1}	N_{-1}^T	$p_e(0.3039635509)$	N_1	N_1^T	$p_e(0.3039635509)$
15200000	4619957	4620246.0	0.3039445395	4620528	4620246.0	0.3039821053
15300000	4650383	4650642.3	0.3039466013	4650883	4650642.3	0.3039792810
15400000	4680596	4681038.7	0.3039348052	4681468	4681038.7	0.3039914286
15500000	4711047	4711435.0	0.3039385161	4711798	4711435.0	0.3039869677
15600000	4741674	4741831.4	0.3039534615	4741957	4741831.4	0.3039716026
15700000	4771945	4772227.7	0.3039455414	4772481	4772227.7	0.3039796815
15800000	4802182	4802624.1	0.3039355696	4803037	4802624.1	0.3039896835
15900000	4832633	4833020.5	0.3039391824	4833375	4833020.5	0.3039858491
16000000	4863066	4863416.8	0.3039416250	4863730	4863416.8	0.3039831250
16100000	4893413	4893813.2	0.3039386957	4894182	4893813.2	0.3039864596
16200000	4923786	4924209.5	0.3039374074	4924602	4924209.5	0.3039877778
16300000	4954368	4954605.9	0.3039489571	4954828	4954605.9	0.3039771779
16400000	4984486	4985002.2	0.3039320732	4985493	4985002.2	0.3039934756
16500000	5014984	5015398.6	0.3039384242	5015787	5015398.6	0.3039870909
16600000	5045530	5045794.9	0.3039475904	5046040	5045794.9	0.3039783133
16700000	5075930	5076191.3	0.3039479042	5076427	5076191.3	0.3039776647
16800000	5106482	5106587.7	0.3039572619	5106673	5106587.7	0.3039686310
16900000	5137006	5136984.0	0.3039648521	5136938	5136984.0	0.3039608284
17000000	5167558	5167380.4	0.3039740000	5167181	5167380.4	0.3039518235
17100000	5197978	5197776.7	0.3039753216	5197540	5197776.7	0.3039497076
17200000	5228079	5228173.1	0.3039580814	5228214	5228173.1	0.3039659302
17300000	5258540	5258569.4	0.3039618497	5258563	5258569.4	0.3039631792
17400000	5289027	5288965.8	0.3039670690	5288881	5288965.8	0.3039586782
17500000	5319647	5319362.1	0.3039798286	5319054	5319362.1	0.3039459429
17600000	5350010	5349758.5	0.3039778409	5349482	5349758.5	0.3039478409
17700000	5380420	5380154.9	0.3039785311	5379862	5380154.9	0.3039470056
17800000	5410822	5410551.2	0.3039787640	5410261	5410551.2	0.3039472472
17900000	5441148	5440947.6	0.3039747486	5440722	5440947.6	0.3039509497
18000000	5471659	5471343.9	0.3039810556	5470992	5471343.9	0.3039440000
18100000	5501925	5501740.3	0.3039737569	5501525	5501740.3	0.3039516575
18200000	5532231	5532136.6	0.3039687363	5532004	5532136.6	0.3039562637
18300000	5562820	5562533.0	0.3039792350	5562211	5562533.0	0.3039459563
18400000	5593139	5592929.3	0.3039749457	5592689	5592929.3	0.3039504891
18500000	5623413	5623325.7	0.3039682703	5623208	5623325.7	0.3039571892
18600000	5653791	5653722.0	0.3039672581	5653631	5653722.0	0.3039586559
18700000	5684183	5684118.4	0.3039670053	5684045	5684118.4	0.3039596257
18800000	5714612	5714514.8	0.3039687234	5714404	5714514.8	0.3039576596
18900000	5745055	5744911.1	0.3039711640	5744769	5744911.1	0.3039560317

Table 3: The comparison of empirical probability and frequency with those from number theory when $\mu(n) = -1/1$ (continued)

N	N_{-1}	N_{-1}^T	$p_e(0.3039635509)$	N_1	N_1^T	$p_e(0.3039635509)$
19000000	5775469	5775307.5	0.3039720526	5775143	5775307.5	0.3039548947
19100000	5805915	5805703.8	0.3039746073	5805503	5805703.8	0.3039530366
19200000	5836335	5836100.2	0.3039757813	5835864	5836100.2	0.3039512500
19300000	5866727	5866496.5	0.3039754922	5866274	5866496.5	0.3039520207
19400000	5897268	5896892.9	0.3039828866	5896543	5896892.9	0.3039455155
19500000	5927598	5927289.2	0.3039793846	5926989	5927289.2	0.3039481538
19600000	5957965	5957685.6	0.3039778061	5957419	5957685.6	0.3039499490
19700000	5988337	5988082.0	0.3039764975	5987834	5988082.0	0.3039509645
19800000	6018757	6018478.3	0.3039776263	6018221	6018478.3	0.3039505556
19900000	6049168	6048874.7	0.3039782915	6048608	6048874.7	0.3039501508
20000000	6079764	6079271.0	0.3039882000	6078811	6079271.0	0.3039405500

Note: N : Length of the block from $\mu(1)$ to $\mu(N)$; N_{-1} : Frequency of $\mu(n) = -1$; $N_{-1}^T (= N \times p_t)$: Frequency of $\mu(n) = -1$ from number theory; $p_e (= \frac{N_{-1}}{N})$: Empirical probability; The number in bracket is theoretical probability from number theory p_t . N_1 and N_1^T are those for $\mu(n) = 1$.

Table 4: The comparison of empirical probability and frequency with those from number theory when $\mu(n) = 0/|\mu(n)| = 1$

N	N_0	N_0^T	p_e (0.3920728981)	$N_{ \mu(n) =1}$	$N_{ \mu(n) =1}^T$	p_e (0.6079271019)
100000	39206	39207.3	0.3920600000	60794	60792.7	0.6079400000
200000	78419	78414.6	0.3920950000	121581	121585.4	0.6079050000
300000	117622	117621.9	0.3920733333	182378	182378.1	0.6079266667
400000	156835	156829.2	0.3920875000	243165	243170.8	0.6079125000
500000	196042	196036.4	0.3920840000	303958	303963.6	0.6079160000
600000	235246	235243.7	0.3920766667	364754	364756.3	0.6079233333
700000	274442	274451.0	0.3920600000	425558	425549.0	0.6079400000
800000	313658	313658.3	0.3920725000	486342	486341.7	0.6079275000
900000	352869	352865.6	0.3920766667	547131	547134.4	0.6079233333
1000000	392074	392072.9	0.3920740000	607926	607927.1	0.6079260000
1100000	431273	431280.2	0.3920663636	668727	668719.8	0.6079336364
1200000	470491	470487.5	0.3920758333	729509	729512.5	0.6079241667
1300000	509697	509694.8	0.3920746154	790303	790305.2	0.6079253846
1400000	548909	548902.1	0.3920778571	851091	851097.9	0.6079221429
1500000	588109	588109.3	0.3920726667	911891	911890.7	0.6079273333
1600000	627308	627316.6	0.3920675000	972692	972683.4	0.6079325000
1700000	666520	666523.9	0.3920705882	1033480	1033476.1	0.6079294118
1800000	705724	705731.2	0.3920688889	1094276	1094268.8	0.6079311111
1900000	744923	744938.5	0.3920647368	1155077	1155061.5	0.6079352632
2000000	784123	784145.8	0.3920615000	1215877	1215854.2	0.6079385000
2100000	823350	823353.1	0.3920714286	1276650	1276646.9	0.6079285714
2200000	862554	862560.4	0.3920700000	1337446	1337439.6	0.6079300000
2300000	901781	901767.7	0.3920786957	1398219	1398232.3	0.6079213043
2400000	940978	940975.0	0.3920741667	1459022	1459025.0	0.6079258333
2500000	980186	980182.2	0.3920744000	1519814	1519817.8	0.6079256000
2600000	1019393	1019389.5	0.3920742308	1580607	1580610.5	0.6079257692
2700000	1058604	1058596.8	0.3920755556	1641396	1641403.2	0.6079244444
2800000	1097812	1097804.1	0.3920757143	1702188	1702195.9	0.6079242857
2900000	1137018	1137011.4	0.3920751724	1762982	1762988.6	0.6079248276
3000000	1176227	1176218.7	0.3920756667	1823773	1823781.3	0.6079243333
3100000	1215433	1215426.0	0.3920751613	1884567	1884574.0	0.6079248387
3200000	1254637	1254633.3	0.3920740625	1945363	1945366.7	0.6079259375
3300000	1293842	1293840.6	0.3920733333	2006158	2006159.4	0.6079266667
3400000	1333046	1333047.9	0.3920723529	2066954	2066952.1	0.6079276471
3500000	1372244	1372255.1	0.3920697143	2127756	2127744.9	0.6079302857
3600000	1411456	1411462.4	0.3920711111	2188544	2188537.6	0.6079288889
3700000	1450677	1450669.7	0.3920748649	2249323	2249330.3	0.6079251351

Table 4: The comparison of empirical probability and frequency with those from number theory when $\mu(n) = 0/|\mu(n)| = 1$ (continued)

N	N_0	N_0^T	$p_e(0.3920728981)$	$N_{ \mu(n) =1}$	$N_{ \mu(n) =1}^T$	$p_e(0.6079271019)$
3800000	1489867	1489877.0	0.3920702632	2310133	2310123.0	0.6079297368
3900000	1529070	1529084.3	0.3920692308	2370930	2370915.7	0.6079307692
4000000	1568264	1568291.6	0.3920660000	2431736	2431708.4	0.6079340000
4100000	1607483	1607498.9	0.3920690244	2492517	2492501.1	0.6079309756
4200000	1646702	1646706.2	0.3920719048	2553298	2553293.8	0.6079280952
4300000	1685897	1685913.5	0.3920690698	2614103	2614086.5	0.6079309302
4400000	1725111	1725120.8	0.3920706818	2674889	2674879.2	0.6079293182
4500000	1764313	1764328.0	0.3920695556	2735687	2735672.0	0.6079304444
4600000	1803532	1803535.3	0.3920721739	2796468	2796464.7	0.6079278261
4700000	1842741	1842742.6	0.3920725532	2857259	2857257.4	0.6079274468
4800000	1881940	1881949.9	0.3920708333	2918060	2918050.1	0.6079291667
4900000	1921148	1921157.2	0.3920710204	2978852	2978842.8	0.6079289796
5000000	1960367	1960364.5	0.3920734000	3039633	3039635.5	0.6079266000
5100000	1999568	1999571.8	0.3920721569	3100432	3100428.2	0.6079278431
5200000	2038780	2038779.1	0.3920730769	3161220	3161220.9	0.6079269231
5300000	2077999	2077986.4	0.3920752830	3222001	3222013.6	0.6079247170
5400000	2117199	2117193.6	0.3920738889	3282801	3282806.4	0.6079261111
5500000	2156411	2156400.9	0.3920747273	3343589	3343599.1	0.6079252727
5600000	2195609	2195608.2	0.3920730357	3404391	3404391.8	0.6079269643
5700000	2234820	2234815.5	0.3920736842	3465180	3465184.5	0.6079263158
5800000	2274032	2274022.8	0.3920744828	3525968	3525977.2	0.6079255172
5900000	2313237	2313230.1	0.3920740678	3586763	3586769.9	0.6079259322
6000000	2352443	2352437.4	0.3920738333	3647557	3647562.6	0.6079261667
6100000	2391647	2391644.7	0.3920732787	3708353	3708355.3	0.6079267213
6200000	2430860	2430852.0	0.3920741935	3769140	3769148.0	0.6079258065
6300000	2470074	2470059.3	0.3920752381	3829926	3829940.7	0.6079247619
6400000	2509271	2509266.5	0.3920735938	3890729	3890733.5	0.6079264063
6500000	2548477	2548473.8	0.3920733846	3951523	3951526.2	0.6079266154
6600000	2587678	2587681.1	0.3920724242	4012322	4012318.9	0.6079275758
6700000	2626883	2626888.4	0.3920720896	4073117	4073111.6	0.6079279104
6800000	2666073	2666095.7	0.3920695588	4133927	4133904.3	0.6079304412
6900000	2705295	2705303.0	0.3920717391	4194705	4194697.0	0.6079282609
7000000	2744496	2744510.3	0.3920708571	4255504	4255489.7	0.6079291429
7100000	2783707	2783717.6	0.3920714085	4316293	4316282.4	0.6079285915
7200000	2822922	2822924.9	0.3920725000	4377078	4377075.1	0.6079275000
7300000	2862131	2862132.2	0.3920727397	4437869	4437867.8	0.6079272603
7400000	2901347	2901339.4	0.3920739189	4498653	4498660.6	0.6079260811

Table 4: The comparison of empirical probability and frequency with those from number theory when $\mu(n) = 0/|\mu(n)| = 1$ (continued)

N	N_0	N_0^T	$p_e(0.3920728981)$	$N_{ \mu(n) =1}$	$N_{ \mu(n) =1}^T$	$p_e(0.6079271019)$
7500000	2940564	2940546.7	0.3920752000	4559436	4559453.3	0.6079248000
7600000	2979774	2979754.0	0.3920755263	4620226	4620246.0	0.6079244737
7700000	3018974	3018961.3	0.3920745455	4681026	4681038.7	0.6079254545
7800000	3058176	3058168.6	0.3920738462	4741824	4741831.4	0.6079261538
7900000	3097382	3097375.9	0.3920736709	4802618	4802624.1	0.6079263291
8000000	3136597	3136583.2	0.3920746250	4863403	4863416.8	0.6079253750
8100000	3175802	3175790.5	0.3920743210	4924198	4924209.5	0.6079256790
8200000	3215009	3214997.8	0.3920742683	4984991	4985002.2	0.6079257317
8300000	3254220	3254205.1	0.3920746988	5045780	5045794.9	0.6079253012
8400000	3293421	3293412.3	0.3920739286	5106579	5106587.7	0.6079260714
8500000	3332635	3332619.6	0.3920747059	5167365	5167380.4	0.6079252941
8600000	3371849	3371826.9	0.3920754651	5228151	5228173.1	0.6079245349
8700000	3411052	3411034.2	0.3920749425	5288948	5288965.8	0.6079250575
8800000	3450264	3450241.5	0.3920754545	5349736	5349758.5	0.6079245455
8900000	3489452	3489448.8	0.3920732584	5410548	5410551.2	0.6079267416
9000000	3528658	3528656.1	0.3920731111	5471342	5471343.9	0.6079268889
9100000	3567878	3567863.4	0.3920745055	5532122	5532136.6	0.6079254945
9200000	3607080	3607070.7	0.3920739130	5592920	5592929.3	0.6079260870
9300000	3646279	3646278.0	0.3920730108	5653721	5653722.0	0.6079269892
9400000	3685477	3685485.2	0.3920720213	5714523	5714514.8	0.6079279787
9500000	3724672	3724692.5	0.3920707368	5775328	5775307.5	0.6079292632
9600000	3763892	3763899.8	0.3920720833	5836108	5836100.2	0.6079279167
9700000	3803083	3803107.1	0.3920704124	5896917	5896892.9	0.6079295876
9800000	3842293	3842314.4	0.3920707143	5957707	5957685.6	0.6079292857
9900000	3881506	3881521.7	0.3920713131	6018494	6018478.3	0.6079286869
10000000	3920709	3920729.0	0.3920709000	6079291	6079271.0	0.6079291000
10100000	3959908	3959936.3	0.3920700990	6140092	6140063.7	0.6079299010
10200000	3999113	3999143.6	0.3920699020	6200887	6200856.4	0.6079300980
10300000	4038336	4038350.9	0.3920714563	6261664	6261649.1	0.6079285437
10400000	4077543	4077558.1	0.3920714423	6322457	6322441.9	0.6079285577
10500000	4116746	4116765.4	0.3920710476	6383254	6383234.6	0.6079289524
10600000	4155966	4155972.7	0.3920722642	6444034	6444027.3	0.6079277358
10700000	4195171	4195180.0	0.3920720561	6504829	6504820.0	0.6079279439
10800000	4234373	4234387.3	0.3920715741	6565627	6565612.7	0.6079284259
10900000	4273572	4273594.6	0.3920708257	6626428	6626405.4	0.6079291743
11000000	4312782	4312801.9	0.3920710909	6687218	6687198.1	0.6079289091
11100000	4352000	4352009.2	0.3920720721	6748000	6747990.8	0.6079279279

Table 4: The comparison of empirical probability and frequency with those from number theory when $\mu(n) = 0/|\mu(n)| = 1$ (continued)

N	N_0	N_0^T	$p_e(0.3920728981)$	$N_{ \mu(n) =1}$	$N_{ \mu(n) =1}^T$	$p_e(0.6079271019)$
11200000	4391205	4391216.5	0.3920718750	6808795	6808783.5	0.6079281250
11300000	4430410	4430423.7	0.3920716814	6869590	6869576.3	0.6079283186
11400000	4469607	4469631.0	0.3920707895	6930393	6930369.0	0.6079292105
11500000	4508834	4508838.3	0.3920725217	6991166	6991161.7	0.6079274783
11600000	4548033	4548045.6	0.3920718103	7051967	7051954.4	0.6079281897
11700000	4587245	4587252.9	0.3920722222	7112755	7112747.1	0.6079277778
11800000	4626443	4626460.2	0.3920714407	7173557	7173539.8	0.6079285593
11900000	4665647	4665667.5	0.3920711765	7234353	7234332.5	0.6079288235
12000000	4704860	4704874.8	0.3920716667	7295140	7295125.2	0.6079283333
12100000	4744060	4744082.1	0.3920710744	7355940	7355917.9	0.6079289256
12200000	4783279	4783289.4	0.3920720492	7416721	7416710.6	0.6079279508
12300000	4822484	4822496.6	0.3920718699	7477516	7477503.4	0.6079281301
12400000	4861696	4861703.9	0.3920722581	7538304	7538296.1	0.6079277419
12500000	4900904	4900911.2	0.3920723200	7599096	7599088.8	0.6079276800
12600000	4940126	4940118.5	0.3920734921	7659874	7659881.5	0.6079265079
12700000	4979326	4979325.8	0.3920729134	7720674	7720674.2	0.6079270866
12800000	5018527	5018533.1	0.3920724219	7781473	7781466.9	0.6079275781
12900000	5057731	5057740.4	0.3920721705	7842269	7842259.6	0.6079278295
13000000	5096933	5096947.7	0.3920717692	7903067	7903052.3	0.6079282308
13100000	5136150	5136155.0	0.3920725191	7963850	7963845.0	0.6079274809
13200000	5175363	5175362.3	0.3920729545	8024637	8024637.7	0.6079270455
13300000	5214577	5214569.5	0.3920734586	8085423	8085430.5	0.6079265414
13400000	5253788	5253776.8	0.3920737313	8146212	8146223.2	0.6079262687
13500000	5292984	5292984.1	0.3920728889	8207016	8207015.9	0.6079271111
13600000	5332181	5332191.4	0.3920721324	8267819	8267808.6	0.6079278676
13700000	5371392	5371398.7	0.3920724088	8328608	8328601.3	0.6079275912
13800000	5410605	5410606.0	0.3920728261	8389395	8389394.0	0.6079271739
13900000	5449803	5449813.3	0.3920721583	8450197	8450186.7	0.6079278417
14000000	5489028	5489020.6	0.3920734286	8510972	8510979.4	0.6079265714
14100000	5528220	5528227.9	0.3920723404	8571780	8571772.1	0.6079276596
14200000	5567421	5567435.2	0.3920719014	8632579	8632564.8	0.6079280986
14300000	5606628	5606642.4	0.3920718881	8693372	8693357.6	0.6079281119
14400000	5645851	5645849.7	0.3920729861	8754149	8754150.3	0.6079270139
14500000	5685061	5685057.0	0.3920731724	8814939	8814943.0	0.6079268276
14600000	5724258	5724264.3	0.3920724658	8875742	8875735.7	0.6079275342
14700000	5763472	5763471.6	0.3920729252	8936528	8936528.4	0.6079270748
14800000	5802667	5802678.9	0.3920720946	8997333	8997321.1	0.6079279054

Table 4: The comparison of empirical probability and frequency with those from number theory when $\mu(n) = 0/|\mu(n)| = 1$ (continued)

N	N_0	N_0^T	$p_e(0.3920728981)$	$N_{ \mu(n) =1}$	$N_{ \mu(n) =1}^T$	$p_e(0.6079271019)$
14900000	5841894	5841886.2	0.3920734228	9058106	9058113.8	0.6079265772
15000000	5881111	5881093.5	0.3920740667	9118889	9118906.5	0.6079259333
15100000	5920310	5920300.8	0.3920735099	9179690	9179699.2	0.6079264901
15200000	5959515	5959508.1	0.3920733553	9240485	9240491.9	0.6079266447
15300000	5998734	5998715.3	0.3920741176	9301266	9301284.7	0.6079258824
15400000	6037936	6037922.6	0.3920737662	9362064	9362077.4	0.6079262338
15500000	6077155	6077129.9	0.3920745161	9422845	9422870.1	0.6079254839
15600000	6116369	6116337.2	0.3920749359	9483631	9483662.8	0.6079250641
15700000	6155574	6155544.5	0.3920747771	9544426	9544455.5	0.6079252229
15800000	6194781	6194751.8	0.3920747468	9605219	9605248.2	0.6079252532
15900000	6233992	6233959.1	0.3920749686	9666008	9666040.9	0.6079250314
16000000	6273204	6273166.4	0.3920752500	9726796	9726833.6	0.6079247500
16100000	6312405	6312373.7	0.3920748447	9787595	9787626.3	0.6079251553
16200000	6351612	6351580.9	0.3920748148	9848388	9848419.1	0.6079251852
16300000	6390804	6390788.2	0.3920738650	9909196	9909211.8	0.6079261350
16400000	6430021	6429995.5	0.3920744512	9969979	9970004.5	0.6079255488
16500000	6469229	6469202.8	0.3920744848	10030771	10030797.2	0.6079255152
16600000	6508430	6508410.1	0.3920740964	10091570	10091589.9	0.6079259036
16700000	6547643	6547617.4	0.3920744311	10152357	10152382.6	0.6079255689
16800000	6586845	6586824.7	0.3920741071	10213155	10213175.3	0.6079258929
16900000	6626056	6626032.0	0.3920743195	10273944	10273968.0	0.6079256805
17000000	6665261	6665239.3	0.3920741765	10334739	10334760.7	0.6079258235
17100000	6704482	6704446.6	0.3920749708	10395518	10395553.4	0.6079250292
17200000	6743707	6743653.8	0.3920759884	10456293	10456346.2	0.6079240116
17300000	6782897	6782861.1	0.3920749711	10517103	10517138.9	0.6079250289
17400000	6822092	6822068.4	0.3920742529	10577908	10577931.6	0.6079257471
17500000	6861299	6861275.7	0.3920742286	10638701	10638724.3	0.6079257714
17600000	6900508	6900483.0	0.3920743182	10699492	10699517.0	0.6079256818
17700000	6939718	6939690.3	0.3920744633	10760282	10760309.7	0.6079255367
17800000	6978917	6978897.6	0.3920739888	10821083	10821102.4	0.6079260112
17900000	7018130	7018104.9	0.3920743017	10881870	10881895.1	0.6079256983
18000000	7057349	7057312.2	0.3920749444	10942651	10942687.8	0.6079250556
18100000	7096550	7096519.5	0.3920745856	11003450	11003480.5	0.6079254144
18200000	7135765	7135726.7	0.3920750000	11064235	11064273.3	0.6079250000
18300000	7174969	7174934.0	0.3920748087	11125031	11125066.0	0.6079251913
18400000	7214172	7214141.3	0.3920745652	11185828	11185858.7	0.6079254348
18500000	7253379	7253348.6	0.3920745405	11246621	11246651.4	0.6079254595

Table 4: The comparison of empirical probability and frequency with those from number theory when $\mu(n) = 0/|\mu(n)| = 1$ (continued)

N	N_0	N_0^T	p_e (0.3920728981)	$N_{ \mu(n) =1}$	$N_{ \mu(n) =1}^T$	p_e (0.6079271019)
18600000	7292578	7292555.9	0.3920740860	11307422	11307444.1	0.6079259140
18700000	7331772	7331763.2	0.3920733690	11368228	11368236.8	0.6079266310
18800000	7370984	7370970.5	0.3920736170	11429016	11429029.5	0.6079263830
18900000	7410176	7410177.8	0.3920728042	11489824	11489822.2	0.6079271958
19000000	7449388	7449385.1	0.3920730526	11550612	11550614.9	0.6079269474
19100000	7488582	7488592.4	0.3920723560	11611418	11611407.6	0.6079276440
19200000	7527801	7527799.6	0.3920729687	11672199	11672200.4	0.6079270313
19300000	7566999	7567006.9	0.3920724870	11733001	11732993.1	0.6079275130
19400000	7606189	7606214.2	0.3920715979	11793811	11793785.8	0.6079284021
19500000	7645413	7645421.5	0.3920724615	11854587	11854578.5	0.6079275385
19600000	7684616	7684628.8	0.3920722449	11915384	11915371.2	0.6079277551
19700000	7723829	7723836.1	0.3920725381	11976171	11976163.9	0.6079274619
19800000	7763022	7763043.4	0.3920718182	12036978	12036956.6	0.6079281818
19900000	7802224	7802250.7	0.3920715578	12097776	12097749.3	0.6079284422
20000000	7841425	7841458.0	0.3920712500	12158575	12158542.0	0.6079287500

Note: N : Length of the block from $\mu(1)$ to $\mu(N)$; N_0 : Frequency of $\mu(n) = 0$; $N_0^T (= N \times p_t)$: Frequency of $\mu(n) = 0$ from number theory; $p_e (= \frac{N_0}{N})$: Empirical probability; The number in bracket is theoretical probability from number theory p_t . $N_{|\mu(n)|=1}$ and $N_{|\mu(n)|=1}^T$ are those for $|\mu(n)| = 1$.

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